Transformation of a System of Independent Variables.

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Let $x, y, z, \ldots.$ be any number of independent variables; $u, v, w, \ldots.$ the same number of independent functions of $x, y, z, \ldots.$ and ϕ any function of $u, v, w, \ldots.$; being therefore also a function of $x, y, z, \ldots.$ We wish to transform any differentiation of ϕ with regard to $x, y, z, \ldots.$ into differentiations of ϕ with regard to $u, v, w, \ldots.$ or in other words to obtain for any differential coefficient $\left(\frac{d}{dx}\right)^r\left(\frac{d}{dy}\right)^s\left(\frac{d}{dz}\right)^t \ldots. \phi$ its equivalent expression of the form $\sum A_{\lambda, \mu, \nu \ldots.}\left(\frac{d}{du}\right)^\lambda\left(\frac{d}{dv}\right)^\mu\left(\frac{d}{dw}\right)^\nu \ldots. \phi$, where $A_{\lambda, \mu, \nu \ldots.}$ is a function of the differential coefficients of $u, v, w, \ldots.$ with respect to $x, y, z, \ldots.$ and independent of the form of ϕ. First supposing ϕ and u to be functions of but one independent variable x, we will have $\left(\frac{d}{dx}\right)^r \phi = \sum A_\lambda\left(\frac{d}{du}\right)^\lambda \phi$, an equivalence which may also be stated in the form $\left(\frac{d}{dx}\right)^r = \sum A_\lambda\left(\frac{d}{du}\right)^\lambda$, where A_λ is a function of differential coefficients of u with regard to x and independent of the form of ϕ. Assume that, α being an arbitrary parameter, we have in any manner obtained an expression for the r^{th} derivative of $\rho^{\alpha u}$ in the form $\left(\frac{d}{dx}\right)^r \rho^{\alpha u} = \sum B_\rho \alpha^\rho . \rho^{\alpha u}$, where B_ρ is independent of α. Since $\rho^{\alpha u}$ is a particular form of $\phi(u)$, we will have also evidently $\left(\frac{d}{dx}\right)^\rho \rho^{\alpha u} = \sum A_\lambda\left(\frac{d}{du}\right)^\lambda \rho^{\alpha u} = \sum A_\lambda \alpha^\lambda . \rho^{\alpha u}$, where A_λ is the same as in the general case, and being independent of the form of ϕ is in this case independent of α. The two expressions for the r^{th} derivative of $\rho^{\alpha u}$ must be equal and therefore $\sum A_\lambda \alpha^\lambda . \rho^{\alpha u} = \sum B_\rho \alpha^\rho . \rho^{\alpha u}$. Consequently

$\sum A_\lambda \alpha^\lambda = \sum B_\rho \alpha^\rho$, and since this equation holds for any value of α while also the coefficients A and B do not involve α, we must have $A_\lambda \equiv B_\lambda$. We have then $\left(\frac{d}{dx}\right)^r = \sum A_\lambda \left(\frac{d}{du}\right)^\lambda = \sum B_\lambda \left(\frac{d}{du}\right)^\lambda$. In order now to find the general formula for the transformation of the r^{th} derivative of ϕ, we first find the r^{th} derivative with regard to x of $\rho^{\alpha u}$ in the form $\sum B_\lambda \alpha^\lambda . \rho^{\alpha u}$, and substituting $\frac{d}{du}$ for α in the summation of $\sum B_\lambda \alpha^\lambda$ obtain thence a formula which holds in general—

$$\left(\frac{d}{dx}\right)^r = \sum B_\lambda \left(\frac{d}{du}\right)^\lambda. \tag{1}$$

We proceed to find the r^{th} derivative of $\rho^{\alpha u}$. The coefficient of ξ^r in the expression of a function $F(x+\xi)$ in powers of ξ is $\frac{1}{r!}\left(\frac{d}{dx}\right)^r F(x)$; consequently $\frac{1}{r!}\left(\frac{d}{dx}\right)^r \rho^{\alpha u}$ is equal to the coefficient of ξ^r in the expansion in powers of ξ of $\rho^{\alpha\left(u + u'\xi + \frac{u''}{2!}\xi^2 \dots\right)}$

$$= \rho^{\alpha u} . \rho^{\alpha u'\xi} . \rho^{\alpha \frac{u''}{2!}\xi^2} \dots . \rho^{\alpha \frac{u^{(\kappa)}}{\kappa!}\xi^\kappa} \dots = \rho^{\alpha u} \sum \frac{(u'\xi)^{p_1} \left(\frac{u''}{2!}\xi^2\right)^{p_2} \dots \left(\frac{u^{(\kappa)}}{\kappa!}\xi^\kappa\right)^{p_\kappa} \dots}{p_1!\, p_2! \dots p_\kappa \dots} \alpha^{p_1 + p_2 + \dots}$$

We have therefore

$$\frac{1}{r!}\left(\frac{d}{dx}\right)^r \rho^{\alpha u} = \rho^{\alpha u} \sum \frac{(u')^{p_1} \left(\frac{u''}{2!}\right)^{p_2} \dots \left(\frac{u^{(\kappa)}}{\kappa!}\right)^{p_\kappa} \dots}{p_1!\, p_2! \dots p_\kappa! \dots} \alpha^{p_1 + p_2 + \dots} \equiv \frac{1}{r!} \rho^{\alpha u} \sum B_\lambda \alpha^\lambda \tag{2}$$

where the summation includes all terms such that $p_1 + 2p_2 + \dots + \kappa p_\kappa + \dots = r$.

Replacing α by $\frac{d}{du}$ we hence obtain from formula (1),

$$\frac{1}{r!}\left(\frac{d}{dx}\right)^r = \sum \frac{(u')^{p_1} (u'')^{p_2} \dots (u^{(\kappa)})^{p_\kappa} \dots}{p_1!\, p_2!\, (2!)^{p_2} \dots p_\kappa!\, (\kappa!)^{p_\kappa} \dots} \left(\frac{d}{du}\right)^\lambda \tag{3}$$

where $\lambda = p_1 + p_2 + \dots p_\kappa + \dots$, the summation being subject as before to the conditions $p_1 + 2p_2 + \dots + \kappa p_\kappa + \dots = r$. This theorem was first obtained by Faà de Bruno,* who proves it by induction. Proofs have also been given by M. Bertrand† and by M. de Presle.‡ Formula (3) may be neatly expressed in

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* Annales de Tortolini, 1855; reproduced in Théorie de Formes Binaires, p. 304.

† Cal. Diff., p. 308.

‡ Bulletin de la Société Mathématique, Vol. XVI, p. 157.

the form of a symbolic determinant. On differentiating $(r-1)$ times in succession the differential equation $\frac{dy}{dx}-\frac{dz}{dx}y=0$ of which a solution is $y=\rho^z$, and solving for $\frac{d^ry}{dxr}$, we obtain in the form of a determinant an expression for the r^{th} derivative of ρ^z, which may be written

$$\left(\frac{d}{dx}\right)^r\rho^z=\begin{vmatrix} z', & \frac{z''}{1!}, & \frac{z'''}{2!}, \dots & \frac{z^{(r)}}{(r-1)!} \\ -(r-1), & z', & \frac{z''}{1!}, \dots & \frac{z^{(r-1)}}{(r-2)!} \\ 0, & -(r-2), & z', \dots & \frac{z^{(r-2)}}{(r-3)!} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 \dots & -1, z' \end{vmatrix}\rho^z$$

In this expression substituting αu for z and afterwards $\frac{d}{du}$ for α, we will have by (1),

$$\left(\frac{d}{dx}\right)^r=\begin{vmatrix} u'\frac{d}{du}, & \frac{u''}{1!}\frac{d}{du}, & \frac{u'''}{2!}\frac{d}{du}\dots & \frac{u^{(r)}}{(r-1)!}\frac{d}{du} \\ -(r-1), & u'\frac{d}{du}, & \frac{u''}{1!}\frac{d}{du}\dots & \frac{u^{(r-1)}}{(r-2)!}\frac{d}{du} \\ 0, & -(r-2), & u'\frac{d}{du}\dots & \frac{u^{(r-2)}}{(r-3)!}\frac{d}{du} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 \dots & -1, u'\frac{d}{du} \end{vmatrix}=$$

$$\begin{vmatrix} u', & \frac{u''}{1!}, & \frac{u'''}{2!}\dots & \frac{u^{(r)}}{(r-1)!} \\ -(r-1)\left(\frac{d}{du}\right)^{-1}, & u', & \frac{u''}{1!}\dots & \frac{u^{(r-1)}}{(r-2)!} \\ 0, & -(r-2)\left(\frac{d}{du}\right)^{-1}, & u'\dots & \frac{u^{(r-2)}}{(r-3)!} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0\dots & -\left(\frac{d}{du}\right)^{-1}, u' \end{vmatrix}\left(\frac{d}{du}\right)^r \qquad (4)$$

where of course $\frac{d}{du}$ is supposed not to operate upon any element of the determinant itself, a supposition which in the case of the second form of determinant may be taken account of by operating first in any term with those elements which involve $\left(\frac{d}{du}\right)^{-1}$.

We might also notice a generalization of the well-known theorem $\left(\frac{d}{du}\right)^r \rho^{\alpha u} v = \rho^{\alpha u}\left(\frac{d}{du}+\alpha\right)^r v$. For brevity writing $\Delta_r\left(\frac{d}{du}\right)$ instead of the second determinant of (4) we have—

$$\left(\frac{d}{dx}\right)^r \rho^{\alpha u} v = \Delta_r\left(\frac{d}{du}\right)\cdot\left(\frac{d}{du}\right)^r\cdot\rho^{\alpha u} v = \rho^{\alpha u}\Delta_r\left(\frac{d}{du}+\alpha\right)\cdot\left(\frac{d}{du}+\alpha\right)^r\cdot v =$$

$$\rho^{\alpha u}\begin{vmatrix} u', & \frac{u''}{1!}, & \frac{u'''}{2!} \cdots\cdots & \frac{u^{(r)}}{(r-1)!} \\ -(r-1)\left(\frac{d}{du}+\alpha\right)^{-1}, & u', & \frac{u''}{1!}\cdots\cdots & \frac{u^{(r-1)}}{(r-2)} \\ 0-(r-2)\left(\frac{d}{du}+\alpha\right)^{-1}, & u', & \cdots\cdots & \frac{u^{(r-2)}}{(r-3)!} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0\ \cdots\cdots -\left(\frac{d}{du}+\alpha\right)^{-1}, & u' \end{vmatrix}\left(\frac{d}{du}+\alpha\right)^r\cdot v \quad (5)$$

Our method of procedure in dealing with a function of several variables will be the same as that employed in the case of one variable. If ϕ be a function of any number of variables $u, v, w, \ldots.$ each of which is a function of the same number of independent variables $x, y, z, \ldots.$ we will have $\left(\frac{d}{dx}\right)^r\left(\frac{d}{dy}\right)^s \ldots.\phi = \sum A_{\lambda,\mu,\ldots.}\left(\frac{d}{du}\right)^\lambda\left(\frac{d}{dv}\right)^\mu\ldots.\phi$, an expression of finite degree in $\frac{d}{du}, \frac{d}{dv}, \ldots.$ the coefficients A of which are functions of the differential coefficients of $u, v, \ldots.$ with regard to $x, y \ldots.$, and independent of the form of ϕ. Suppose we have in any manner obtained a formula

$$\left(\frac{d}{dx}\right)^r\left(\frac{d}{dy}\right)^s\ldots.\rho^{\alpha u+\beta v+\ldots.} = \sum B_{\rho,m,\ldots.}\alpha^\rho\beta^m\ldots.\rho^{\alpha u+\beta v+\ldots.}$$

where the coefficients B are independent of the arbitrary parameters $\alpha, \beta, \ldots.$ Since $\rho^{\alpha u+\beta v+\cdots}$ is a particular form of ϕ, we will have also

$$\left(\frac{d}{dx}\right)^r\left(\frac{d}{dy}\right)^s\ldots\rho^{\alpha u+\beta v+\cdots}=\sum A_{\lambda,\mu,\ldots}\left(\frac{d}{du}\right)^\lambda\left(\frac{d}{dv}\right)^\mu\ldots\rho^{\alpha u+\beta v+\cdots}=$$

$\sum A_{\lambda,\mu,\ldots}\,\alpha^\lambda\beta^\mu\ldots\rho^{\alpha u+\beta v+\cdots}$ where the coefficients A are independent of the parameters $\alpha, \beta, \ldots.$ Consequently

$$\sum A_{\lambda,\mu}\ldots\alpha^\lambda\beta^\mu\ldots=\sum B_{\rho,m,\ldots}\,\alpha^\rho\beta^m\ldots$$

and the functional coefficients A and B not involving the arbitrary parameters, we will have $A_{\lambda,\mu\ldots}=B_{\lambda,\mu\ldots}$, and hence

$$\sum A_{\lambda,\mu,\ldots}\left(\frac{d}{du}\right)^\lambda\left(\frac{d}{dv}\right)^\mu\ldots=\sum B_{\lambda,\mu,\ldots}\left(\frac{d}{du}\right)^\lambda\left(\frac{d}{dv}\right)^\mu\ldots$$

Our process therefore will be to first obtain the formula

$$\left(\frac{d}{dx}\right)^r\left(\frac{d}{dy}\right)^s\ldots\rho^{\alpha u+\beta v\cdots}=\sum B_{\lambda,\mu\ldots}\,\alpha^\lambda\beta^\mu\ldots\rho^{\alpha u+\beta v+\cdots}$$

then substitute $\frac{d}{du}, \frac{d}{dv}\ldots$ for $\alpha, \beta, \ldots$ respectively in the summation $\sum B_{\lambda,\mu}\ldots\alpha^\lambda\beta^\mu\ldots$ obtaining thence the formula

$$\left(\frac{d}{dx}\right)^r\left(\frac{d}{dy}\right)^s\ldots=\sum B_{\lambda,u}\ldots\left(\frac{d}{du}\right)^\lambda\left(\frac{d}{dv}\right)^\mu\ldots \tag{6}$$

which, as we have seen, must hold good in general.

We will now determine the form of the coefficients B in (6).

We know that $\frac{1}{r!}\left(\frac{d}{dx}\right)^r\frac{1}{s!}\left(\frac{d}{dy}\right)^s\ldots F(x, y, \ldots)$ is the coefficient of $\xi^r\eta^s\ldots$ in the expansion of $F(x+\xi, y+\eta, \ldots)$ in powers of $\xi, \eta, \ldots.$ If then ρ^ω is a function of $x, y, \ldots$ we will have $\frac{1}{r!\;s!\ldots}\left(\frac{d}{dx}\right)^r\left(\frac{d}{dy}\right)^s\ldots\rho^\omega$ for the coefficient of $\xi^r\eta^s\ldots$ in the expansion of $\rho^{\omega+\Delta\omega+\frac{\Delta^2}{2!}\omega+\cdots}=\rho^{\rho^\Delta\omega}$ where $\Delta\equiv\xi\frac{d}{dx}+\eta\frac{d}{dy}+\ldots.$ Putting $\omega\equiv\alpha u+\beta v+\ldots$, we have

$$\frac{1}{r!\;s!\ldots}\left(\frac{d}{dx}\right)^r\left(\frac{d}{dy}\right)^s\ldots\rho^{\alpha u+\beta v+\cdots}=\text{coefficient of }\xi^r\eta^s\ldots\text{ in }\rho^{\rho^\Delta(\alpha u+\beta v+\cdots)} \tag{7}$$

$$\text{Now } \rho^{\rho^{\Delta}(\alpha u+\beta v+\ \ldots)} = \rho^{\rho^{\Delta}\alpha u} \cdot \rho^{\rho^{\Delta}\beta v} \ldots\ldots = \rho^{\left(\rho^{\xi\frac{d}{dx}},\ \rho^{\eta\frac{d}{dy}} \ldots\right)\alpha u} \cdot \rho^{\rho^{\Delta}\beta v} \ldots\ldots$$

$$= \rho^{\sum\frac{\left(\xi\frac{d}{dx}\right)^a}{a!} \cdot \sum\frac{\left(\eta\frac{d}{dy}\right)^b}{b!} \ldots\ldots \alpha u} \cdot \rho^{\rho^{\Delta}\beta v} \ldots\ldots = \rho^{\sum\frac{\left(\xi\frac{d}{dx}\right)^a}{a!}\frac{\left(\eta\frac{d}{dy}\right)^b}{b!}\frac{\ldots\ldots}{\ldots\ldots}\alpha u} \cdot \rho^{\rho^{\Delta}\beta v} \ldots\ldots$$

$$= \rho^{\alpha u} \cdot \rho^{\frac{\left(\xi\frac{d}{dx}\right)^{a_1}}{a_1!}\frac{\left(\eta\frac{d}{dy}\right)^{b_1}}{b_2!}\frac{\ldots\ldots}{\ldots\ldots}\alpha u} \cdot \rho^{\frac{\left(\xi\frac{d}{dx}\right)^{a_2}}{a_2!}\frac{\left(\eta\frac{d}{dy}\right)^{b_2}}{b_2!}\frac{\ldots\ldots}{\ldots}\alpha u} \ldots\ldots \rho^{\rho^{\Delta}\beta v} \ldots\ldots$$

(where in none of the exponents of ρ are all the numerical exponents a, b, ... zero at the same time)

$$= \rho^{\alpha u} \sum \frac{1}{p_1!\ p_2! \ldots\ldots} \left(\frac{\left(\xi\frac{d}{dx}\right)^{a_1}}{a_1!}\frac{\left(\eta\frac{d}{dy}\right)^{b_1}}{b_1!}\frac{\ldots\ldots\alpha u}{\ldots\ldots}\right)^{p_1} \cdot \left(\frac{\left(\xi\frac{d}{dx}\right)^{a_2}}{a_2!}\frac{\left(\eta\frac{d}{dy}\right)^{2}}{b_2!}\frac{\ldots\alpha u}{\ldots\ldots}\right)^{p_2} \ldots\ldots \rho^{\rho^{\Delta}\beta v} \ldots\ldots$$

Developing $\rho^{\rho^{\Delta}\beta v}$, $\rho^{\rho^{\Delta}\gamma w}$ .... in the same way, we find

$$\rho^{\rho^{\Delta}\omega} = \rho^{\omega} \sum \frac{1}{\pi(p!)\ \pi(p'!)\ \pi(p''!) \ldots\ldots} \left(\frac{\left(\xi\frac{d}{dx}\right)^{a_1}}{a_1!}\frac{\left(\eta\frac{d}{dy}\right)^{b_1}}{b_1!}\frac{\ldots\ldots\alpha u}{\ldots\ldots}\right)^{p_1}\left(\frac{\left(\xi\frac{d}{dx}\right)^{a_2}}{a_2!}\frac{\left(\eta\frac{d}{dy}\right)^{b_2}}{b_2!}\frac{\ldots\ldots\alpha u}{\ldots\ldots}\right)^{p_2} \ldots\ldots$$
$$\left(\frac{\left(\xi\frac{d}{dx}\right)^{a'_1}}{a'_1!}\frac{\left(\eta\frac{d}{dy}\right)^{b'_1}}{b'_1!}\frac{\ldots\ldots\beta v}{\ldots\ldots}\right)^{p'_1}\left(\frac{\left(\xi\frac{d}{dx}\right)^{a'_2}}{a'_2!}\frac{\left(\eta\frac{d}{dy}\right)^{b'_2}}{b'_2!}\frac{\ldots\ldots\beta v}{\ldots\ldots}\right)^{p'_2} \ldots\ldots \tag{8}$$

where for brevity the product $p_1!\ p_2! \ldots\ldots$ is represented by $\pi(p!) \ldots\ldots$

We will now have the coefficient of $\xi^r\eta^s \ldots\ldots$ in $\rho^{\rho^{\Delta}\omega}$ equal to

$$\frac{1}{r!\ s! \ldots\ldots}\left(\frac{d}{dx}\right)^r\left(\frac{d}{dy}\right)^s \ldots\ldots \rho^{\alpha u+\beta v+\ \ldots\ldots} = \rho^{\omega}\sum\frac{1}{\pi(p!)\ \pi(p'!) \ldots\ldots}$$
$$\left(\frac{\left(\frac{d}{dx}\right)^{a_1}}{a_1!}\frac{\left(\frac{d}{dy}\right)^{b_1}}{b_1!}\frac{\ldots\ldots u}{\ldots\ldots}\right)^{p_1}\left(\frac{\left(\frac{d}{dx}\right)^{a_2}}{a_2!}\frac{\left(\frac{d}{dy}\right)^{b}}{b_2!}\frac{\ldots u}{\ldots\ldots}\right)^{p_2} \ldots \left(\frac{\left(\frac{d}{dx}\right)^{a'_1}}{a'_1!}\frac{\left(\frac{d}{dy}\right)^{b'_1}}{b'_1!}\frac{\ldots v}{\ldots\ldots}\right)^{p'_1}\left(\frac{\left(\frac{d}{dx}\right)^{a'_2}}{a'_2!}\frac{\left(\frac{d}{dy}\right)^{b'_2}}{b'_2!}\frac{\ldots\ldots v}{\ldots\ldots}\right)^{p'_2} \ldots \alpha^{\Sigma p} \cdot \beta^{\Sigma p'} \ldots \tag{9}$$

subject to the conditions $\Sigma pa + \Sigma p'a' + \ldots\ldots = r$, $\Sigma pb + \Sigma p'b' + \ldots\ldots = s$, etc.

We may write (9) in the form

$$\frac{1}{r!\ s! \ldots\ldots}\left(\frac{d}{dx}\right)^r\left(\frac{d}{dy}\right)^s \ldots\ldots \rho^{\alpha u+\beta v+\ \ldots\ldots}$$
$$= \sum C\pi(\nabla u)^p \cdot \pi(\nabla v)^{p'} \ldots\ldots \alpha^{\lambda}\beta^{\mu} \ldots\ldots \rho^{\alpha u+\beta v+\ \ldots\ldots} \tag{10}$$

where by ∇u is meant any differential coefficient of u with regard to x, y, ... and where $\pi(\nabla u)^p$ designates any product of powers of differential coefficient of u, C.... being a constant coefficient for any term and equal in value to the reciprocal of the product of the factorials of all the exponents which appear in the term, each exponent being taken as often as it appears in the term, *e. g.* in the term which appears explicitly in (9), a_1 enters as an exponent p_1 times while p_1 occurs once, and consequently in the denominator of the coefficient

$a_1!$ appears p_1 times and $p_1!$ once. The only limitations upon the summation in (10), as we see from (9), are that the dimensions of each term in $\left(\frac{d}{dx}\right), \left(\frac{d}{dy}\right), \ldots.$ must respectively equal $r, s, \ldots.$ while $\lambda, \mu, \ldots.$ must indicate its dimensions in $u, v, \ldots.$ respectively. From (10) by (6) we now derive

$$\frac{1}{r!\, s! \ldots.} \left(\frac{d}{dx}\right)^r \left(\frac{d}{dy}\right)^s \ldots. = \sum C\pi(\nabla u)^p . \pi(\nabla v)^{p'} \ldots. \left(\frac{d}{du}\right)^\lambda \left(\frac{d}{dv}\right)^\mu \ldots. \tag{11}$$

where to repeat ourself the only conditions upon the summation are:

I. The dimensions of every term in $\frac{d}{dx}, \frac{d}{dy}, \ldots.$ respectively must be $r, s, \ldots.$

II. The dimensions of every term in $u, v, \ldots.$ respectively must be zero.

III. The coefficient in any term is the reciprocal of the product of the factorials of all exponents appearing in the term, each being taken as often as it occurs.

That all the terms which occur in the transformed expression for $\left(\frac{d}{dx}\right)^r \left(\frac{d}{dy}\right)^s \ldots.$ must satisfy the first two conditions is from the beginning self-evident, but that all the products which satisfy these conditions do occur is not so evident, nor is it immediately apparent what the values of the numerical coefficients will be.